

Laplacian State Transfer in Coronas

Ethan Ackelsberg* Zachary Brehm† Ada Chan‡ Joshua Mundinger§
Christino Tamon¶

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Abstract

We prove that the corona product of two graphs has no Laplacian perfect state transfer whenever the first graph has at least two vertices. This complements a result of Coutinho and Liu who showed that no tree of size greater than two has Laplacian perfect state transfer. In contrast, we prove that the corona product of two graphs exhibits Laplacian pretty good state transfer, under some mild conditions. This provides the first known examples of families of graphs with Laplacian pretty good state transfer. Our result extends of the work of Fan and Godsil on double stars to the Laplacian setting. Moreover, we also show that the corona product of any cocktail party graph with a single vertex graph has Laplacian pretty good state transfer, even though odd cocktail party graphs have no perfect state transfer.

1 Introduction

Given a graph G and a symmetric matrix M associated with G , the continuous-time quantum walk on G relative to M is given by the unitary matrix

$$U(t) := \exp(-itM). \quad (1)$$

This notion was introduced by Farhi and Gutmann [11] as a paradigm to design efficient quantum algorithms. Physically, this also represents the evolution of a quantum spin system. This interesting connection was explored in the works of Bose [3] and Christandl *et al.* [6, 5]. Furthermore, as pointed out by Bose *et al.* [4], there are two different matrices M of interest. In the so-called XY model, M is the adjacency matrix of G ; in the XYZ model, M is the Laplacian of G . For details on these physical models, see Bose *et al.* [4] for a derivation. Note that if G is regular, these quantum walks differ only by complex conjugation and a phase factor.

From the physical standpoint, quantum walks relative to the adjacency matrix and the Laplacian are equally important. However, the current literature has focused mostly on quantum walks relative to the adjacency matrix. In this paper, we investigate continuous-time quantum walks relative to the Laplacian.

We are interested in the phenomenon of state transfer, which models the routing of information between particles in the associated spin system. This was the original motivation of the work by

*Division of Science, Mathematics and Computing, Bard College at Simon's Rock.

†Department of Mathematics, SUNY Potsdam.

‡Department of Mathematics and Statistics, York University.

§Department of Mathematics and Statistics, Swarthmore College.

¶Department of Computer Science, Clarkson University.

Bose [3]. A graph G is said to have perfect state transfer between vertices u and v if there is a time t such that

$$|\exp(-itM)_{uv}|^2 = 1. \quad (2)$$

Physically, this means that the probability of state transfer between vertices u and v is unity. We will refer to the matrix entry $\exp(-itM)_{uv}$ as the transition element between the vertices u and v .

There are several infinite families of graphs known to have perfect state transfer. This includes hypercubes [5], some families of distance-regular graphs [8], complete graphs with a missing edge [4], and some joins [1]. However, recently it has become clear that perfect state transfer is rare. In the adjacency matrix case, Godsil showed that there are at most finitely many graphs with a given maximum valency with perfect state transfer [14], while, in the Laplacian case, Coutinho and Liu showed that there is no perfect state transfer on trees with at least three vertices [9].

Nevertheless, transmission of information in a quantum system may not occur perfectly, but rather with probability that is arbitrarily close to unity. We thus consider a relaxation. A graph G is said to have pretty good state transfer if for each $\epsilon > 0$, there exists a time t such that

$$|\exp(-itM)_{uv}|^2 \geq 1 - \epsilon. \quad (3)$$

This notion was proposed by Godsil [13]. Relative to the adjacency matrix, pretty good state transfer was studied on paths [15] and double stars [10]. However, prior to this work, there are no known families of graphs with pretty good state transfer in Laplacian quantum walks. In this paper, we provide the first infinite families of graphs with Laplacian pretty good state transfer. Our families of graphs are constructed using the corona product of two graphs.

The corona product was introduced by Frucht and Harary to construct a graph whose automorphism group is the wreath product of the automorphism groups of the component graphs [12]. Quantum walks (in a discrete-time setting) on the corona of hypercubes and cliques were studied numerically by Makmal *et al.* [18]. In this work, we show that there is no perfect state transfer on coronas, continuing to support that perfect state transfer is rare. However, we show that given a graph G with perfect state transfer, the corona product of G with another graph H will have pretty good state transfer, subject to a condition on the number of vertices of H . As a corollary, we extend the work of Fan and Godsil on double stars [10] to the Laplacian, as well as construct an infinite family of coronas with pretty good state transfer where neither component of the product has any state transfer.

For more information on algebraic graph theory, see Godsil and Royle [16], and for a survey on state transfer on graphs, see Godsil [13].

2 Preliminaries

Let G be a graph with adjacency matrix A and diagonal degree matrix D . The *Laplacian* of G is defined as the matrix $L := D - A$. The unitary matrix

$$U(t) := \exp(-itL) \quad (4)$$

determines a *continuous-time Laplacian quantum walk* on G . We are interested in studying perfect state transfer and pretty good state transfer in such a quantum walk. If G has n vertices, we associate the vertices of G with coordinates in \mathbb{C}^n , and let $|u\rangle$ denote the characteristic vector of the vertex u .

A graph G has *perfect state transfer* between vertices u and v at time τ if there exists a complex number γ such that

$$U(\tau) |u\rangle = \gamma |v\rangle. \quad (5)$$

The complex number γ is called the *phase* of the perfect state transfer. Since $U(t)$ is unitary for all t , the condition (5) is equivalent to $|U(\tau)_{uv}|^2 = 1$. Since L is real symmetric, $U(t)$ is symmetric, which shows that (5) is symmetric in u and v , that is, $U(\tau)|v\rangle = \gamma|u\rangle$.

By the spectral theorem, L admits a decomposition

$$L = \sum_{i=0}^d \lambda_i F_{\lambda_i}, \quad (6)$$

where $0 = \lambda_0 < \lambda_1 < \dots < \lambda_d$ are the distinct eigenvalues of L and F_{λ_i} is the eigenprojector for eigenvalue λ_i . Note that $\sum_{i=0}^d F_{\lambda_i} = I$.

We say two vertices u and v are *strongly cospectral* if, for every eigenvalue λ of G , we have

$$F_{\lambda}|u\rangle = \pm F_{\lambda}|v\rangle. \quad (7)$$

The *eigenvalue support* of a vertex v is the set of all eigenvalues λ of G such that $F_{\lambda}|v\rangle \neq 0$.

The following theorem states the known necessary and sufficient conditions for Laplacian perfect state transfer.

Theorem 2.1 (Coutinho [7], Theorem 7.3.1). *Let G be a graph, and let u and v be vertices in G . Let S be the eigenvalue support of u . Then, there is Laplacian perfect state transfer between u and v at time τ if and only if all of the following hold:*

- (i) *The vertices u and v are strongly cospectral;*
- (ii) *All eigenvalues in S are integers;*
- (iii) *For each $\lambda \in S$, $\langle u|F_{\lambda}|v\rangle$ is positive if and only if $\lambda/\gcd(S)$ is even.*

Moreover, if these hold, there is a minimum time of perfect state transfer given by $t_0 := \pi/\gcd(S)$, and τ is an odd multiple of t_0 .

A graph G has *pretty good state transfer* between vertices u and v if for every $\epsilon > 0$, there exists a time t and there exists a complex number γ such that

$$\|U(t)|u\rangle - \gamma|v\rangle\| < \epsilon. \quad (8)$$

Since $U(t)$ is unitary, this is equivalent to requiring that for each $\epsilon > 0$, there exists a time t such that

$$|U(t)_{uv}|^2 > 1 - \epsilon. \quad (9)$$

As in Godsil *et al.* [15], our main tool for showing the existence of pretty good state transfer is Kronecker's Approximation Theorem.

Theorem 2.2 (Hardy and Wright [17], Theorem 442). *Let $1, \lambda_1, \dots, \lambda_m$ be linearly independent over \mathbb{Q} . Let $\alpha_1, \dots, \alpha_m$ be arbitrary real numbers, and let ϵ be a positive real number. Then, there is an integer ℓ and integers q_1, \dots, q_m so that*

$$|\ell\lambda_k - \alpha_k - q_k| < \epsilon, \quad (10)$$

for each $k = 1, \dots, m$.

For brevity, whenever we have an equation of the form $|\alpha - \beta| < \epsilon$, we will write instead $\alpha \approx \beta$ and omit the explicit dependence on ϵ . For example, (10) will be represented as $\ell\lambda_k - q_k \approx \alpha_k$.

In our applications of Kronecker's Theorem, we will use the following lemma to identify sets of numbers which are linearly independent over the rationals.

Lemma 2.3 (Richards [19]). *The set $\{\sqrt{\Delta} : \Delta \text{ is a square-free integer}\}$ is linearly independent over the set of rational numbers \mathbb{Q} .*

Notation: Let $|j_m\rangle$ denote the all-one vector with m components, and let $J_{m,n}$ denote the $m \times n$ all-one matrix (or simply J_m if $m = n$). In cases where we need to specify the underlying graph, we will use a notation such as $L(G)$ instead of L . Throughout this paper, the spectrum of G is the spectrum of its Laplacian L , denoted $\text{Sp}(G)$, unless stated otherwise.

3 Corona of Graphs

Let G be a graph on the vertex set $V(G) = \{v_1, \dots, v_n\}$ and let $\vec{H} = (H_1, \dots, H_n)$ be an n -tuple of graphs. The (*inhomogenous*) *corona* $G \circ \vec{H}$ is formed by taking the disjoint union of G and H_1, \dots, H_n and then adding an edge from each vertex in H_j to the vertex v_j in G . Formally, the corona $G \circ \vec{H}$ has the vertex set

$$V(G \circ \vec{H}) = \{(v, 0) : v \in V(G)\} \cup \bigcup_{j=1}^n \{(v_j, w) : v_j \in V(G), w \in V(H_j)\} \quad (11)$$

and the adjacency relation

$$(v_j, w) \sim (v_k, w') \iff \begin{cases} w = w' = 0 \text{ and } v_j \sim_G v_k, & \text{or} \\ v_j = v_k \text{ and } w \sim_{H_j} w', & \text{or} \\ v_j = v_k \text{ and exactly one of } w \text{ and } w' \text{ is } 0. \end{cases} \quad (12)$$

For notational convenience, we will identify $V(G)$ with $\{1, 2, \dots, n\}$, whereby we simply denote v_ℓ by ℓ .

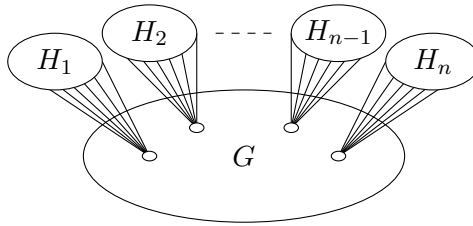


Figure 1: An inhomogeneous corona $G \circ \vec{H}$ where $\vec{H} = (H_1, \dots, H_n)$.

The adjacency matrix of the corona $G \circ \vec{H}$ is given by

$$A(G \circ \vec{H}) = A(G) \otimes |0\rangle\langle 0| + \sum_{\ell=1}^n |\ell\rangle\langle \ell| \otimes \begin{bmatrix} 0 & \langle j_m | \\ |j_m\rangle & A(H_\ell) \end{bmatrix}. \quad (13)$$

Then, it follows that the Laplacian of $G \circ \vec{H}$ is

$$L(G \circ \vec{H}) = (L(G) + mI) \otimes |0\rangle\langle 0| + \sum_{\ell=1}^n |\ell\rangle\langle \ell| \otimes \begin{bmatrix} 0 & -\langle j_m | \\ -|j_m\rangle & L(H_\ell) + I \end{bmatrix}. \quad (14)$$

The definition given by Frucht and Harary [12] coincides with when \vec{H} is a constant sequence (H, \dots, H) for some graph H . We will denote such a corona with a constant sequence simply as $G \circ H$. In this case, the spectrum of $G \circ H$ is known.

Theorem 3.1 (Barik *et al.* [2], Theorem 3.2). *Let G be a graph on n vertices and H be a graph on m vertices. Suppose G has spectrum $0 = \lambda_0 < \dots < \lambda_p$ with multiplicities r_0, \dots, r_p , and H has spectrum $0 = \mu_0 < \mu_1 < \dots < \mu_q$ with multiplicities r'_0, \dots, r'_q . Then the (homogeneous) corona $G \circ H$ has the following spectrum:*

- (a) 1 with multiplicity $n(r'_0 - 1)$;
- (b) $\mu_j + 1$ with multiplicity nr'_j , for each $j = 1, \dots, q$;
- (c) $\frac{1}{2} \left(\lambda_j + m + 1 \pm \sqrt{(\lambda_j + m + 1)^2 - 4\lambda_j} \right)$ with multiplicity r_j , for each $j = 1, \dots, p$.

In the following, we extend Theorem 3.1 by computing the eigenvalues and eigenprojectors of the inhomogenous corona $G \circ (H_1, \dots, H_n)$ when the order $|V(H_j)|$ is the same for each $j = 1, \dots, n$.

Proposition 3.2. *Let G be a graph on n vertices and $\vec{H} = (H_1, \dots, H_n)$ be an n -tuple of graphs on $m \geq 1$ vertices. Then the spectrum of $G \circ \vec{H}$ is given by:*

- (a) 1 is an eigenvalue, if there exists H_ℓ that is disconnected, with the eigenprojector

$$F_1 = \sum_{\ell=1}^n |\ell\rangle\langle\ell| \otimes \begin{bmatrix} 0 & \langle 0_m | \\ |0_m\rangle & F_0(H_\ell) - \frac{1}{m}J_m \end{bmatrix}. \quad (15)$$

Note $F_0(H_j) = \frac{1}{m}J_m$ if and only if H_j is connected.

- (b) $\mu + 1$ is an eigenvalue, if there exists H_ℓ that has a nonzero eigenvalue μ , with the eigenprojector

$$F_{\mu+1} = \sum_{\ell=1}^n |\ell\rangle\langle\ell| \otimes \begin{bmatrix} 0 & \langle 0_m | \\ |0_m\rangle & F_\mu(H_\ell) \end{bmatrix}, \quad (16)$$

where we assume $F_\mu(H_\ell) = 0$ if μ is not an eigenvalue of H_ℓ .

- (c) $\lambda_\pm = \frac{1}{2}(m + \lambda + 1 \pm \sqrt{(m + \lambda - 1)^2 + 4m})$ are eigenvalues, for each eigenvalue λ of G , with eigenprojectors

$$F_{\lambda_\pm} = \frac{1}{(1 - \lambda_\pm)^2 + m} F_\lambda(G) \otimes \begin{bmatrix} (1 - \lambda_\pm)^2 & (1 - \lambda_\pm) \langle j_m | \\ (1 - \lambda_\pm) |j_m\rangle & J_m \end{bmatrix}. \quad (17)$$

Therefore, the spectral decomposition of the Laplacian of the corona $G \circ \vec{H}$ is given by

$$L(G \circ \vec{H}) = \sum_{\lambda \in \text{Sp}(G)} \sum_{\pm} \lambda_\pm F_{\lambda_\pm} + \sum_{\mu} (\mu + 1) F_{\mu+1}, \quad (18)$$

where the sum over μ is over all eigenvalues of the graphs H_ℓ , for $\ell = 1, \dots, n$.

Proof. Let H_ℓ be one of the graphs in the sequence \vec{H} . Suppose $|x\rangle$ is a normalized eigenvector of $L(H_\ell)$ corresponding to eigenvalue μ , and that $|x\rangle$ is orthogonal to $|j_m\rangle$. Then,

$$L(G \circ \vec{H}) |\ell\rangle \otimes \begin{bmatrix} 0 \\ |x\rangle \end{bmatrix} = (\mu + 1) |\ell\rangle \otimes \begin{bmatrix} 0 \\ |x\rangle \end{bmatrix}. \quad (19)$$

If H_ℓ is disconnected, then $L(H_\ell)$ has eigenvectors with eigenvalue zero orthogonal to $|j_m\rangle$. In this case, 1 is an eigenvalue of $L(G \circ \vec{H})$ with the eigenprojector given in (15). For each eigenvalue $\mu \neq 0$ of $L(H_\ell)$ with eigenprojector $F_\mu(H_\ell)$, $\mu + 1$ is an eigenvalue of $L(G \circ \vec{H})$ with the eigenprojector given in (16).

Since $|V(G)| = n$, then (15) and (16) together give $n(m - 1)$ eigenvectors of $L(G \circ \vec{H})$. We construct the remaining $2n$ eigenvectors using the eigenvectors of $L(G)$. Suppose $|y\rangle$ is an eigenvector of $L(G)$ with eigenvalue λ . Then,

$$L(G \circ \vec{H}) |y\rangle \otimes \begin{bmatrix} 1 - \lambda_\pm \\ |j_m\rangle \end{bmatrix} = \lambda_\pm |y\rangle \otimes \begin{bmatrix} 1 - \lambda_\pm \\ |j_m\rangle \end{bmatrix} \quad (20)$$

if and only if

$$\lambda_\pm = \frac{m + \lambda + 1 \pm \sqrt{(m + \lambda + 1)^2 - 4\lambda}}{2} = \frac{m + \lambda + 1 \pm \sqrt{(m + \lambda - 1)^2 + 4m}}{2}. \quad (21)$$

After normalizing, the eigenprojectors corresponding to eigenvalues λ_\pm are given by (17). \square

Note that when $\vec{H} = (H, \dots, H)$, Proposition 3.2 coincides with Theorem 3.1.

4 Perfect State Transfer

In the following theorem, we show that there is no perfect state transfer between any pair of vertices in the corona of two graphs whenever the first graph has at least two vertices. The proof relies on the fact that there are no integer eigenvalues in the support of any vertex.

Theorem 4.1. *Let G be a connected graph on $n \geq 2$ vertices and $\vec{H} = (H_1, \dots, H_n)$ be an n -tuple of graphs on $m \geq 1$ vertices. Then there is no Laplacian perfect state transfer in $G \circ \vec{H}$.*

Proof. Let (v, w) be a vertex of $G \circ \vec{H}$. By Theorem 2.1, it suffices to find a non-integer eigenvalue in the support of (v, w) . Since G is connected on at least two vertices, there exists a positive eigenvalue λ in the eigenvalue support of v . From Proposition 3.2, both

$$\lambda_\pm = \frac{1}{2} \left(m + \lambda + 1 \pm \sqrt{(m + \lambda - 1)^2 + 4m} \right) \quad (22)$$

are in the eigenvalue support of (v, w) . Suppose towards contradiction that both λ_\pm are integers. Then both

$$m + \lambda + 1 = \lambda_+ + \lambda_-, \quad (23)$$

$$\sqrt{(m + \lambda - 1)^2 + 4m} = \lambda_+ - \lambda_- \quad (24)$$

are integers, implying that λ is an integer. Further, $(m + \lambda - 1)^2 + 4m$ is a perfect square, but since $4m$ is even, the parity of this square must be the same as $(m + \lambda - 1)^2$. Since $\lambda > 0$, the least square greater than $(m + \lambda - 1)^2$ with the same parity is $(m + \lambda + 1)^2$. This yields the bound

$$(m + \lambda - 1)^2 + 4m \geq (m + \lambda + 1)^2 \quad (25)$$

which implies $\lambda \leq 0$, a contradiction. Thus, one of λ_\pm is not an integer. \square

5 Pretty Good State Transfer

Although Theorem 4.1 shows there is no perfect state transfer on coronas, we will show that there is pretty good state transfer, under some mild conditions. First, we state a useful form for the transition elements of a Laplacian quantum walk on coronas.

Proposition 5.1. *Let G be a graph on n vertices and let $\vec{H} = (H_1, \dots, H_n)$ be an n -tuple of graphs on $m \geq 1$ vertices. If u and v are vertices of G , then the transition element between vertices $(u, 0)$ and $(v, 0)$ in $G \circ \vec{H}$ is given by*

$$\begin{aligned} & \langle u, 0 | \exp(-itL(G \circ \vec{H})) | v, 0 \rangle \\ &= e^{-it(m+1)/2} \sum_{\lambda \in \text{Sp}(G)} e^{-it\lambda/2} \langle u | F_\lambda(G) | v \rangle \left(\cos\left(\frac{t}{2}\Delta_\lambda\right) - \frac{(m+\lambda-1)}{\Delta_\lambda} i \sin\left(\frac{t}{2}\Delta_\lambda\right) \right), \end{aligned} \quad (26)$$

where $\Delta_\lambda := \sqrt{(m+\lambda-1)^2 + 4m}$, for each eigenvalue λ of G .

Proof. For each eigenvalue λ of $L(G)$, recall

$$\lambda_\pm = \frac{1}{2}(m + \lambda + 1 \pm \Delta_\lambda). \quad (27)$$

By Proposition 3.2, the transition element between vertices $(u, 0)$ and $(v, 0)$ is given by

$$\langle u, 0 | e^{-itL(G \circ H)} | v, 0 \rangle = \sum_{\lambda \in \text{Sp}(G)} e^{-i(m+\lambda+1)t/2} \langle u | F_\lambda(G) | v \rangle \left(\sum_{\pm} e^{\mp i\Delta_\lambda t/2} \left(\frac{(1-\lambda_\pm)^2}{(1-\lambda_\pm)^2 + m} \right) \right). \quad (28)$$

Given the following identities hold:

$$\prod_{\pm} (1 - \lambda_\pm) = -m, \quad (29a)$$

$$\prod_{\pm} ((1 - \lambda_\pm)^2 + m) = m\Delta_\lambda^2, \quad (29b)$$

we see that

$$\sum_{\pm} e^{\mp i\Delta_\lambda t/2} \left(\frac{(1-\lambda_\pm)^2}{(1-\lambda_\pm)^2 + m} \right) = \cos\left(\frac{t}{2}\Delta_\lambda\right) - \frac{(m+\lambda-1)}{\Delta_\lambda} i \sin\left(\frac{t}{2}\Delta_\lambda\right). \quad (30)$$

This proves the claim. \square

The following theorem shows that if G has perfect state transfer, then $G \circ \vec{H}$ has pretty good state transfer for infinite families of graphs \vec{H} .

Theorem 5.2. *Let G be a graph on n vertices and $\vec{H} = (H_1, \dots, H_n)$ be an n -tuple of graphs on $m \geq 1$ vertices. Suppose G has perfect state transfer between vertices u and v , and let 2^r be the greatest power of two dividing each element of the eigenvalue support of u . If 2^{r+1} divides $m+1$, then there is pretty good state transfer between vertices $(u, 0)$ and $(v, 0)$ in $G \circ \vec{H}$.*

Proof. Let S be the eigenvalue support of u in G . By Theorem 2.1, we know that the eigenvalues in S are integers. Further, if g is the greatest common divisor of all eigenvalues in the support of u , then perfect state transfer occurs at times that are odd multiples of π/g . Since g is an odd multiple of 2^r , this implies that there is perfect state transfer in G at time $\pi/2^r$. For any integer ℓ , we consider the transition element at times

$$t = (4\ell + 2^{1-r})\pi. \quad (31)$$

From Proposition 5.1, recall that $\Delta_\lambda = \sqrt{(m + \lambda - 1)^2 + 4m}$ for each eigenvalue λ of G . We show that for times of the form (31) for specific choices of ℓ , we have

$$\cos(\Delta_\lambda t/2) \approx 1 \quad (32)$$

for all eigenvalues λ in the support of u . Since 2^{r+1} divides $m + 1$, we observe that

$$\exp(-it(m + 1)/2) = 1. \quad (33)$$

If (32) holds, then by Proposition 5.1, at time t , we have

$$\langle u, 0 | e^{-itL(G \circ \vec{H})} | v, 0 \rangle \approx \sum_{\lambda \in \text{Sp}(G)} \exp(-it\lambda/2) \langle u | F_\lambda(G) | v \rangle = \langle u | e^{-i(t/2)L(G)} | v \rangle. \quad (34)$$

Since $t/2 \equiv \pi/2^r \pmod{2\pi}$, there is perfect state transfer in G between u and v at time $t/2$, which shows that (32) is sufficient for pretty good state transfer in $G \circ \vec{H}$.

Note that Δ_λ squares to an integer if λ is an integer. For each λ in the support of u , let c_λ be the square-free part of Δ_λ^2 ; then $\Delta_\lambda = s_\lambda \sqrt{c_\lambda}$ for some integer s_λ . From the proof of Theorem 4.1 we may see that if $m \geq 1$ and λ is a positive integer, then Δ_λ is irrational; this implies that $c_\lambda > 1$ if $\lambda > 0$. By Lemma 2.3, the disjoint union

$$\{1\} \cup \{\sqrt{c_\lambda} : \lambda \in S, \lambda > 0\} \quad (35)$$

is linearly independent over \mathbb{Q} . By Kronecker's Theorem, we may pick integers ℓ, q_λ such that

$$\ell\sqrt{c_\lambda} - q_\lambda \approx -\frac{\sqrt{c_\lambda}}{2^{r+1}}. \quad (36)$$

If $c_\lambda = c_\mu$ for two distinct eigenvalues λ and μ in the support of u , then $q_\lambda = q_\mu$. Multiplying both sides of (36) by $4s_\lambda$ yields

$$(4\ell + 2^{1-r})\Delta_\lambda \approx 4s_\lambda q_\lambda. \quad (37)$$

Therefore, at $t = (4\ell + 2^{1-r})\pi$, we have $\cos(\Delta_\lambda t/2) \approx 1$ for $\lambda > 0$. To take care of when $\lambda = 0$, note that $\cos(\Delta_0 t/2) = \cos((m + 1)t/2) = 1$ from (33). \square

Theorem 5.2 provides sufficient conditions for the existence of pretty good state transfer in $G \circ \vec{H}$ for many sequences \vec{H} of families of graphs. For example, consider the d -cube Q_d . The d -cube has perfect state transfer at time $\pi/2$ (see Christandl *et al.* [6]). Thus, if $m \equiv 3 \pmod{4}$, then $Q_d \circ \vec{H}$ has pretty good state transfer. This provides a partial continuous-time analog to the results of Makmal *et al.* for discrete-time quantum walks on $Q_d \circ H$ [18].

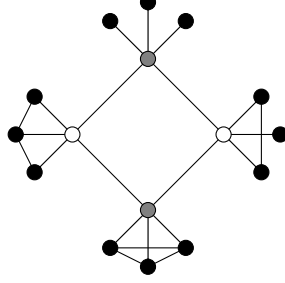


Figure 2: The graph $G \circ \vec{H}$ where \vec{H} is the sequence of non-isomorphic graphs on 3 vertices has pretty good state transfer between the white vertices and between the grey vertices.

In what follows, we show a different way in which Kronecker's Theorem may be applied in conjunction with Proposition 5.1.

Theorem 5.3. *Let $\vec{H} = (H_1, H_2)$ be a pair of graphs on $m \geq 1$ vertices. Then $K_2 \circ \vec{H}$ has pretty good state transfer between the vertices of K_2 .*

Proof. Let u and v denote the vertices of K_2 . The Laplacian of K_2 has eigenvalues 0 and 2, and the corresponding eigenprojectors satisfy $\langle u | F_0(K_2) | v \rangle = 1/2$ and $\langle u | F_2(K_2) | v \rangle = -1/2$. By Proposition 5.1, it is sufficient to approximate the following system of equations:

$$e^{-it(m+3)/2} \cos\left(\frac{t}{2}\sqrt{(m+1)^2 + 4m}\right) \approx -1, \quad (38)$$

$$e^{-it(m+1)/2} \cos\left(\frac{t}{2}(m+1)\right) \approx 1. \quad (39)$$

If $t = 4\pi\ell$ for some integer ℓ , we note that

$$\exp(-it(m+3)/2) = \exp(-it(m+1)/2) = \cos(t(m+1)/2) = 1. \quad (40)$$

Thus, it suffices to show that for $t = 4\pi\ell$, we have

$$\cos\left(\frac{t}{2}\sqrt{(m+1)^2 + 4m}\right) \approx -1. \quad (41)$$

By the proof of Theorem 4.1, we note that $\Delta_2 = \sqrt{(m+1)^2 + 4m}$ is irrational for any positive integer m . By Kronecker's Theorem, we may find integers ℓ and s such that

$$\Delta_2\ell - s \approx 1/2. \quad (42)$$

This implies that $2\pi\ell\Delta_2 \approx 2\pi s + \pi$, so $t = 4\pi\ell$ gives $\cos(t\Delta_2/2) \approx -1$ as desired. \square

Fan and Godsil [10] investigated pretty good state transfer on the double stars $K_2 \circ \overline{K}_m$ relative to the adjacency matrix. They proved that pretty good state transfer relative to the adjacency occurred if and only if $1 + 4m$ is not a perfect square. However, Theorem 5.3 shows that pretty good state transfer on double stars relative to the Laplacian occurs for all m , independent of any number-theoretic conditions on m . Theorem 5.3 also shows that $P_4 = K_2 \circ K_1$ has Laplacian pretty good state transfer, which was originally observed by Godsil.

The following example shows that state transfer in G is not necessary for $G \circ \vec{H}$ to have pretty good state transfer. Coutinho *et al.* [8] proved that the only distance-regular graph of diameter 2 which admits perfect state transfer is $\overline{nK_2}$ for even n . We show that $\overline{nK_2} \circ K_1$ has pretty good state transfer for any $n \geq 2$. First, we state a useful result about distance-regular graphs.

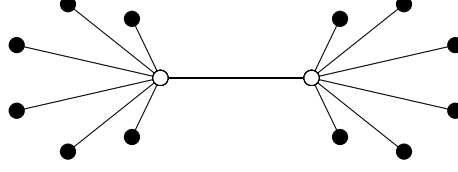


Figure 3: $K_2 \circ \overline{K_6}$ has pretty good state transfer relative to the Laplacian (between the white vertices), but not relative to the adjacency matrix [10].

Lemma 5.4 (Coutinho *et al.* [8], Lemma 4.4). *Let G be a distance-regular graph with diameter d . Suppose G is antipodal with classes of size two and let $\theta_0 > \dots > \theta_d$ be the distinct eigenvalues of $A(G)$ with corresponding eigenprojectors E_0, \dots, E_d . Then, for each $j = 0, \dots, d$, we have*

$$A_d E_j = (-1)^j E_j. \quad (43)$$

Here, A_d is the adjacency matrix of a graph obtained from G by connecting vertices u and v if and only if they are at distance d .

Theorem 5.5. *Let $G = \overline{nK_2}$ be the cocktail party graph on $2n$ vertices for some positive integer $n \geq 2$. Suppose that u and v are antipodal vertices of G . Then there is pretty good state transfer in $G \circ K_1$ between $(u, 0)$ and $(v, 0)$.*

Proof. The cocktail party graph is a distance-regular graph with diameter two and is antipodal with classes of size two. Let the eigenprojectors of $A(G)$ be E_0, E_1 and E_2 . Since G is regular, these are also the eigenprojectors of the Laplacian $L(G)$. The corresponding Laplacian eigenvalues are $\lambda_0 = 0$, $\lambda_1 = 2n - 2$, and $\lambda_2 = 2n$.

Let A_2 be the adjacency matrix of the graph where u and v are adjacent if and only if they are antipodal in G . If u and v are antipodal vertices of G , then by Lemma 5.4,

$$E_j = (-1)^j A_2 E_j \quad (44)$$

for $j = 0, 1, 2$, which implies that

$$\langle u | E_j | v \rangle = (-1)^j \langle u | E_j | u \rangle. \quad (45)$$

Using (26) in Proposition 5.1, we will pick times t such that $e^{-it\lambda_j/2} = 1$ and

$$\cos(\Delta_{\lambda_j} t/2) \approx (-1)^j, \quad (46)$$

where $\Delta_{\lambda_j} = \sqrt{\lambda_j^2 + 4}$. Note by letting $t = 4\pi\ell$, we have $e^{-it\lambda_j/2} = 1$ for all eigenvalues λ_j and $\cos(\Delta_{\lambda_0} t/2) = \cos(t) = 1$. Now $\Delta_{\lambda_1} = 2\sqrt{1 + (n-1)^2}$ and $\Delta_{\lambda_2} = 2\sqrt{1 + n^2}$. Neither of these are integers for $n \geq 2$. Since one of $1 + n^2$ and $1 + (n-1)^2$ is congruent to 2 mod 4 while the other is congruent to 1 mod 4, their square-free parts are distinct. Thus, by Lemma 2.3, the set $\{1, \Delta_{\lambda_1}, \Delta_{\lambda_2}\}$ is linearly independent over \mathbb{Q} . By Kronecker's Theorem, there exist integers ℓ, q_1, q_2 such that

$$\ell \Delta_{\lambda_1} - q_1 \approx 1/2, \quad (47a)$$

$$\ell \Delta_{\lambda_2} - q_2 \approx 0. \quad (47b)$$

This implies that, at $t = 4\pi\ell$, equation (46) is satisfied. \square

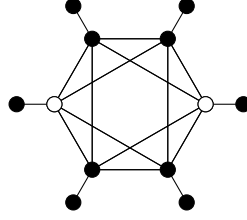


Figure 4: The graph $\overline{3K_2} \circ K_1$ has pretty good state transfer (between the white vertices), while $\overline{3K_2}$ does not have perfect state transfer.

6 Acknowledgments

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